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DESIGN OF FEEDBACK SYSTEMS WITH NONMINIMUM-PHASE UNSTABLE PLANT--ETC(U)
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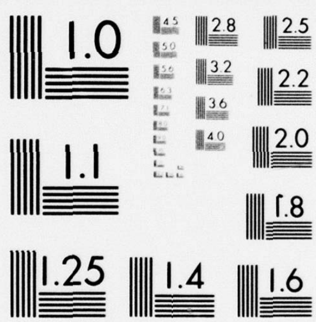
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DESIGN OF FEEDBACK SYSTEMS WITH NONMINIMUM-PHASE UNSTABLE PLANTS

Isaac Horowitz*

Abstract

Feedback systems with right half-plane poles and zeros may have inherently very poor sensitivity properties. In the design procedure presented, the closed-loop poles are restricted to two possible regions in the complex plane. One region is $\mathrm{Re}\, s \le -\sigma$, $\sigma > 0$. A second is the interior and boundary of a circle in the left half-plane. The design is optimum in the sense of maximizing the gain factor uncertainty, for which the restriction is satisfied. The design procedure is very simple to execute and results in loop transmission poles and zeros which are symmetrical with respect to the boundary of the forbidden region. The closed-loop poles lie entirely on the boundary, over the range of gain uncertainty.

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1. INTRODUCTION

Consider the constrained part (plant) of a linear time-invariant feedback system, which because of parameter uncertainty, has a transfer function known only to be a member of a set $\mathcal{P} = \{P(s)\}$. For a large class of minimum-phase (no zeros in the interior of the right half complex plane) $\mathcal{P}$ sets, any narrow but nonzero frequency-domain performance tolerances can be theoretically achieved in a two-degree-of-freedom feedback structure (Horowitz 1963) even for very large but bounded parameter uncertainty. $\mathcal{P}$ may contain elements with uncertain right half-plane (denoted by rhp) poles. A synthesis procedure exists, permitting optimum design to specifications (Horowitz and Sidi 1972). However if $\mathcal{P}$ includes nonminimum-phase elements (i.e. with rhp zeros), then a given set of performance tolerances may not be theoretically achievable $\forall\ P \in \mathcal{P}$. It has been shown how to check if a given specification set is achievable. Also, for a given nonminimum-phase but stable $\mathcal{P}$ set, the problem may be made solvable if it is permitted to decrease sufficiently the system bandwidth (Horowitz and Sidi 1978). Of course, the latter solution may be extremely undesirable, but is unavoidable in the linear time invariant framework. For example, if $\exists$ a $P \in \mathcal{P}$ with a zero at 0.1, the closed-loop bandwidth may have to be a small fraction of 0.1 rps if $\mathcal{P}$ is a 'large' set, in order to achieve reasonable tolerances in the resulting very small system bandwidth. If $\mathcal{P}$ includes both nonminimum-phase (denoted by nmp) and unstable elements, then even the latter is in general unachievable. The sensitivity reduction capabilities of the feedback loop are severely restricted, no matter how small the bandwidth.

This paper considers the nmp unstable $\mathcal{P}$ problem from a somewhat different viewpoint. What is the maximum tolerable plant gain (k)

uncertainty $\rho = k_{max}/k_{min}$, such that the closed-loop poles are restricted to a certain region in the complex plane? One such region considered is $\mathcal{R}e\, s \leq -\sigma$, $\sigma > 0$. Another is the interior and boundary of a circle centered at $-b$ with radius a , $b > a > 0$. What are the trade-offs between σ , b , a and ρ_{max} ? A very simple synthesis procedure is presented for implementing the above. All transfer functions are assumed finite rational functions. Note that pure time-delays can be so approximated as accurately as desired, over any finite frequency range. Since pole-zero uncertainty is not considered, it is assumed that there are no hidden cancellations of rhp poles and zeros in any $P \in \mathcal{P}$.

1.1 One and two-degree-of-freedom structures

A typical tdf (two-degree-of-freedom) structure is shown in Fig. 1. The system transfer function $T(s) = F \frac{L}{(1+L)}$, $L = PG$ and the sensitivity function $S_P^T = \frac{\partial T/T}{\partial P/P} = \frac{1}{1+L}$, are independently realizable to a large extent (Horowitz and Sidi 1972), because two independent functions F , G in $L = GP$ are available to the designer. In the odf (one degree of freedom) system, $F = 1$, so $T = (1-S)$. In the minimum-phase (denoted by mp) system, $|S|$ can at least be designed as small as desired over any finite ω range, making $T \doteq 1$ in this range, i.e. forcing T to have a large bandwidth. In fact, it has been shown that in any practical design $|S| > 1$ in an important frequency range of $T(j\omega)$ (Horowitz 1963).

It will be seen that in nmp unstable systems, $|S|$ cannot in general be made small over any desired ω range. In fact, $|S|$ tends to be embarrasingly large in the range of the system bandwidth. In the tdf structure, one can at least use F to achieve reasonable nominal $|T(j\omega)|$ over any desired frequency range, i.e. at a specific nominal set of plant parameter values. However, in

the odf system where $T = (1-S)$, not only does one have large $|S|$, but also large $|T| \;\forall\; P \in \mathcal{P}$. The design challenge is to do the best possible for S . In the odf structure the design is then complete, while in the tdf there is available F to achieve the desired nomimal $T(s)$. Obviously, the tdf structure is preferable and only requires that any two independent functions of command input r and output c in Fig. 1 be measurable (Horowitz 1963).

1.1.1 Constraints on $S(s)$ and $T(s)$

Since rhp poles and zeros should not be cancelled for well-known reasons, the rhp zeros of P must appear as zeros of T . Most important, the rhp poles and zeros of P must appear as such in $L(s)$. Only these constraints need to be explicitly recognized in our approach. The other constraints (rhp poles of P as zeros of S , etc.) are automatically included.

1.2 Limitations due to rhp poles and zeros

Qualitatively, the limitations on the feedback capabilities of such systems can be explained as follows. If $P(s)$ has a zero at $b > 0$, then the crossover frequency ω_c (defined by $|L(j\omega_c)| = 1$) in any practical design must be $< \alpha b$, $\alpha < 1$ (Horowitz 1963) — assuming that $|L(j\omega)| > 1$ for $\omega < \omega_c$. The latter is, of course, essential to achieve the benefits of feedback in a range $\omega < \omega_c$. On the other hand, if P has a pole at $m > 0$, it is necessary that $\omega_c > \beta m$, $\beta > 1$. How can these two constraints be satisfied if both exist in P with $m > b$, and especially if there are several rhp poles and zeros? It will be seen that a stable design with closed loop poles in the restricted regions of Sec. 1.1 always exists over a finite range of gain uncertainty. The system overcomes the problem by having several crossover frequencies, at least as many as rhp poles. The need for this can be seen also from the Nyquist criterion, which requires as many negative encirclements of -1

as rhp poles. The locus of $L(j\omega)$ then tends to be tightly wound around -1, with resulting very small $|1+L(j\omega)|$ and very large sensitivity function $S(j\omega) = [1+L(j\omega)]^{-1}$, over wide ω intervals.

Consider a typical mp stable practical $L(s) \to ks^{-x}$ with $x \geq 2$ as $s \to \infty$, and $|L(j\omega)| > 1$ in $[0, \omega_c)$. The Nyquist mapping of $L(s)$ (with s encircling the rhp) must encircle $-m$ $(m > 0)$ positively, i.e. in same sense as the s encirclement, usually taken as clockwise with $\omega : 0 \to \infty$. In fact, it is impossible for such $L(s)$ to have a negative encirclement, as seen from the familiar Nyquist formula: Number of encirclements $N = N_z - N_p$. Since N_p the number of rhp poles of L, is 0, $N = N_z$ the number of rhp zeros of L, which must be a non-negative number. But when $N_p > 0$, then negative encirclements are mandatory and are achievable because the rhp poles provide the means, by their contribution of effective phase lead accompanied by amplitude decrease. But one must wait, as ω goes from 0 to ∞, for the rhp pole corner frequency before the combination of phase and amplitude is available for a negative encirclement. That is why in open-loop unstable systems, the crossover frequency ω_c must exceed some minimum value (Horowitz 1963). In the absence of rhp zeros of L, the negative encirclement can be done in any ω range exceeding the minimum, so any desired benefits of feedback are achievable (Horowitz and Sidi 1972), as with mp stable plants. However, a rhp zero corner frequency, by its combination of amplitude increase and effective phase lag, forces the completion of the negative encirclement before the rhp zero effect is too strong — which is associated with a maximum value for ω_c. Thus, in a general sense, lhp poles and zeros tend to give positive encirclements while rhp poles and zeros give negative encirclements. However, lhp poles and zeros can be effectively cancelled and thus easily shifted, postponing the encirclement (where $|L| < 1$) to as high an ω range as desired. Such cancellation

is impossible for rhp poles and zeros. The -1 encirclements and consequent small $|S| = |1+L|^{-1}$ must be taken in the range of occurrence of the rhp poles and zeros. In fact, as will be seen, it may even be necessary to add more of them.

This discussion is, of course, highly qualitative and it is challenging to make it quantitative. The technique presented here does so to a certain extent. It is noted that contributions to this problem have also been made by (Chang 1961, Brasch and Pearson 1970, Shaw 1971, Bongiorno and Yonla 1977). All of these neglect explicit plant uncertainty.

2. SOME RESULTS FROM THE BLENDING PROBLEM

This paper uses some results from the following "blending" problem (Horowitz and Gera 1978): There are given two nmp uncertain plants $P_1 \in \mathcal{P}_1$, $P_2 \in \mathcal{P}_2$, with the same input but whose outputs can be separately measured and processed by G_1, G_2 respectively. Can fixed rational function $G_1(s)$, $G_2(s)$ be found such that the sum $P_1G_1 + P_2G_2$ is mp over $\mathcal{P}_1 \times \mathcal{P}_2$? Equivalently, find H such that $1 + \frac{P_2G_2}{P_1G_1} \triangleq 1 + PH$ (with $P = P_2/P_1$, $H = G_2/G_1$) has no rhp zeros for any $P \in \mathcal{P}_1 \times \mathcal{P}_2$.

If there is uncertainty only in the gain factor k of P (none in its poles and zeros), then the left half-plane (denoted by lhp) poles and zeros of P can be cancelled out by H. Those in the rhp should not, of course, be so eliminated. The optimum PH (defined as that which maximizes $\rho = k_{max}/k_{min}$) has the form $PH = kK\varphi(s)\varphi(-s)$, $\varphi(s)$ monic, so that the poles and zeros of PH are symmetrical with respect to the $j\omega$ axis. Also, PH has an equal number of poles and zeros, which is acceptable because $H = G_2/G_1$. For

$k \in [k_{min}, k_{max}]$ the root loci of $1+PH = 1+k\varphi(s)\varphi(-s)$, all lie on the $j\omega$ axis and cover it completely (see Fig. 2). There is a pair of zeros of $1+PH$ at the origin either at k_{min} or at k_{max} and similarly a pair at $\pm j\infty$. Both at k_{min} and at k_{max} the other zeros of $(1+PH)$ are in coincident pairs on the $j\omega$ axis, i.e. have factors $(s^2+\omega_i^2)^2$, as in Fig. 2. A method was given for finding the optimum PH (which maximizes $k_{max}/k_{min} = \rho$), but a much better method is now presented. The directions are first given together with a numerical example, followed by its derivation.

2.1 Directions for finding optimum PH

Step 1. Only the rhp poles (δ_p in number) and zeros (δ_z in number) of P are explicitly displayed here. If the system is to be type m, let P have m poles at the origin included in the above. The optimum H has $2\delta_p + \delta_z - 2$ zeros and $\delta_p + 2\delta_z - 2$ poles, (not counting those which cancel the lhp poles and zeros of P). Hence, PH has a total $2(\delta_p + \delta_z - 1)$ of poles and of zeros, symmetrical with respect to the $j\omega$ axis, viz $PH = kK\varphi(s)\varphi(-s)$. Accordingly, choose $\varphi(s)$ with only lhp poles and zeros of the required generality, i.e. with $\delta_p - 1$ free (unspecified as yet) zeros, $\delta_z - 1$ free poles and the constrained mirror image of the δ_p rhp poles and the δ_z zeros of P.

Example. $P = \dfrac{k(s-1)(s-4)}{(s-2)(s-3)(s^2-s+4)}$ has $\delta_z = 2$, $\delta_p = 4$ so let

$$\varphi(s) = \frac{n(s)}{d(s)} = \frac{(s+1)(s+4)(s^3+As^2+Bs+C)}{(s+2)(s+3)(s^2+s+4)(s+a)} \quad (1)$$

Step 2. Expand $n(s)$ and $d(s)$ of (1) as follows, with $e_i(s)$, $o_i(s)$ even and odd monic polynomials, respectively.

$$n(s) = o_n(s) + c_n e_n(s) = [s^5 + s^3(B+5A+4) + s(5C+4B)]$$
$$+ (A+5)[s^4 + s^2 \frac{(C+5B+4A)}{A+5} + \frac{4C}{A+5}] \tag{2a}$$

$$d(s) = o_d(s) + c_d e_d(s) = [s^5 + s^3(6a+15) + s(26a+24)]$$
$$+ (a+6)[s^4 + s^2 \frac{(15a+26)}{(a+6)} + \frac{24a}{a+6}] \tag{2b}$$

If $\delta_z + \delta_p - 1$ is even, the expressions each have the form $e_i(s) + c_i o_i(s)$ with degree of e_i = (degree of o_i) + 1 .

Step 3. Make $o_n(s) \equiv o_d(s)$, $e_n(s) \equiv e_d(s)$ by equating coefficients, giving precisely $\delta_z + \delta_p - 2$ equations in as many unknowns. Thus, set

$$B + 5A + 4 = 6a + 15 \quad , \quad 5C + 4B = 26a + 24 \ ,$$

$$\frac{C+5B+4A}{A+5} = \frac{15a+26}{a+6} \quad , \quad \frac{4C}{A+5} = \frac{24a}{a+6} \ , \text{ with solutions}$$

$$a = .66006 \quad , \quad C = 4.14477 \quad , \quad A = 1.97019 \quad , \quad B = 5.10942 \quad ,$$

$$s^3 + As^2 + Bs + C = (s + 1.00133)(s^2 + .96886s + 4.13928)$$

Step 4. $\rho = \frac{k_{max}}{k_{min}} = \left(\frac{A+5}{a+6}\right)^2 = \left(\frac{c_n}{c_d}\right)^2$ of (2a,b) = 1.0953

Step 5. From (2a,b), Step 4 gave $\frac{n}{d}$ in $PH = kK \frac{n(s)n(-s)}{d(s)d(-s)}$. Since P is known, H can be found. The lhp poles and zeros of P , if any, must be added as zeros and poles of H , respectively. The root loci of $1+PH$ are shown in Fig. 2.

Step 6. The zeros of $1+PH$ at k_{min} and k_{max} are available, if desired, from o_i , e_i of Step 2. Thus, $o(s) = s(s^4 + 18.96036s^2 + 41.16155)$ $= s(s^2 + 2.50076)(s^2 + 16.45960)$, so at k_{min} $1+PH$ has zeros at $\pm j\sqrt{2.50076}$, $\pm j\sqrt{16.4596}$. Since $e(s) = (s^2 + 4.90561)(s^2 + .48487)$, at

k_{max} 1+PH has zeros at $\pm j\sqrt{4.90561}$, $\pm j\sqrt{.48487}$.

Derivation

The derivation of the above procedure starts with the results from (Horowitz and Gera 1978) stated in second paragraph of Sec. 2, giving $PH = kK\frac{n(s)n(-s)}{d(s)d(-s)}$ and $\text{Num.}(1+PH) = d(s)d(-s) + kKn(s)n(-s) = (1+Kk)\pi(s^2+\omega_j^2)$ with $\omega_j^2 \geq 0$ for $k \in [k_{min}, k_{max}]$. However, at k_{min} , say

$$d(s)s(-s) + k_{min}Kn(s)n(-s) = K_1 s^2 \pi(s^2+\omega_i^2)^2 \tag{3a}$$

while at k_{max} , say

$$d(s)d(-s) + k_{max}Kn(s)n(-s) = K_2\pi(s^2+\Omega_i^2)^2 \tag{3b}$$

Actually s^2 may appear as a factor in (3b) rather than in (3a) but it is not important which. K_1 , K_2 are used in (3a,b) instead of $1+k_{min}K$, $1+k_{max}K$ because one of the latter must be zero, at which point a pair of zeros of 1+PH is at $\pm j\infty$. Eliminating in turn $d(s)d(-s)$ and $n(s)n(-s)$ from (3a,b) give the pair of equations (with Ω_i^2 , $\omega_i^2 > 0$).

$$(k_{max}-k_{min})Kn(s)n(-s) = K_2\pi(s^2+\Omega_i^2) - K_1 s^2\pi(s^2+\omega_i^2)^2 \tag{4a}$$

$$(k_{min}-k_{max})Kd(s)d(-s) = k_{min}K_2\pi(s^2+\Omega_i^2)^2 - k_{max}K_1 s^2\pi(s^2+\omega_i^2) \tag{4b}$$

Let $n(s) = \pi(s+z_i)$, $d(s) = \pi(s+p_i)$
with $\mathcal{R}e(z_i)$, $\mathcal{R}e(p_i) > 0$. (5a,b)

Replace s^2 by w so that (4a,b) become

$$\gamma_1\pi(w-z_i^2) = \pi(w+\Omega_i^2)^2 - \rho_1^2 w\pi(w+w_i^2)^2$$

$$\gamma_2\pi(w-p_i^2) = \pi(w+\Omega_i^2)^2 - \rho_2^2 w\pi(w+w_i^2)^2 \tag{6a-d}$$

$$\gamma_1 = \frac{(k_{max}-k_{min})K}{K_2} \ , \ \gamma_2 = \frac{(k_{min}-k_{max})K}{k_{min}K_2}$$

$$\rho_1^2 = \frac{K_1}{K_2} \quad , \quad \rho_2^2 = \frac{k_{max}}{k_{min}} \frac{K_1}{K_2} \quad , \quad \rho = \frac{\rho_2^2}{\rho_1^2} = \frac{k_{max}}{k_{min}} \tag{6e-g}$$

Each of (6a,b) has precisely the form of the optimum solution for active RC synthesis by means of "negative impedance conversion (NIC)" (Horowitz 1959). There the problem is to write any given real coefficient polynomial $\psi(w)$, as the difference of two polynomials $A(w)$, $B(w)$ where zeros are all negative real, $\psi(w) = A(w) - \alpha B(w)$. The minus α can be implemented by means of a "negative impedance converter". Of the infinitude of $A(w)$, $B(w)$ available for the task, it is desired to choose that which minimized the sensitivity of $\psi(j\omega)$ to variations in α, $S_\alpha^\psi(j\omega) = \frac{\partial\psi/\psi}{\partial\alpha/\alpha}$. It was shown that the unique optimum $A(w)$, $B(w)$ have precisely the form on the right of (6a,b). Thus $n(s)n(-s)$ and $d(s)d(-s)$ with $w = s^2$ have identical "optimum NIC decomposition" polynomials, differing only in the gain factors.

(Calahan 1960) found a very elegant technique for deriving the optimum NIC decomposition polynomials $A(w)$, $B(w)$ from a given $\psi(w)$. Start with $\psi(w) = \pi(w - r_i^2) = A(w) - \alpha B(w)$ and write $N(u) = \pi(u + r_i)$ with $-r_i$ the <u>left half-plane</u> root of r_i^2. Expand $N(u)$ into even and odd polynomials.

$$N(u) = e_N(u) + o_N(u) = \pi(u^2 + a_i) + \rho u \pi(u^2 + b_i) \quad , \quad a_i, b_i > 0 \; , \tag{7a}$$

because $N(u)$ is Hurwitz (Weinberg 1962). Finally, recover $\psi(u)$ by writing

$$\psi(u) = [e_N^2(u) - o_N^2(u)]_{u^2=w} = \prod_1^n (w+a_i)^2 - \rho^2 w(w+b_i)^2 \tag{7b}$$

with the right side the desired optimum NIC decomposition.

It follows from the above that if $\psi_1(w) = \pi(w - z_i^2)$ of (5a), $\psi_2(w) = \pi(w - p_i^2)$ of (5b), have the identical optimum NIC decomposition monic polynomials, their corresponding $N_1(u)$, $N_2(u)$ in the Calahan technique also

have identical even and odd monic decomposition polynomials. The $n(s)$, $d(s)$ of (5a,b) and Steps 1,2 of the "Directions" (Eqs. 1,2) are precisely the N_i in the above of $n(s)n(-s)$, $d(s)d(-s)$. Hence, their respective monic even and odd decompositions must be identical. In the Calahan technique, the ρ cf N of (7a) is squared to recover ψ in (7b). Hence from (6g), $\rho = k_{max}/k_{min}$ is obtained by means of Step 4 of the "Directions". Note also that the even-odd polynomial decomposition of $n(s)$ (or of $d(s)$), gives the roots of $1+PH$ at k_{min}, k_{max}. The even polynomial gives one and the odd gives the other, as illustrated in Step 6 above.

3. IMPLEMENTATION OF OPTIMUM BLENDING SOLUTION

In the blending problem, the boundary of the undesirable region is the $j\omega$ axis. In the control problem, it is realistic to use as boundary the vertical line $s=-\sigma$, $\sigma>0$, especially as this permits exploitation of the optimum blending results. Clearly, it is only necessary to shift the $j\omega$ axis σ to the left and shift it back at the end. This is illustrated by Example 2.

3.1 Example 2

$P = \frac{k(s-1)}{s(s-2)}$ and the system is to be Type 1, so the pole at the origin is counted as a constrained right half plane pole. Replace s by $v+\sigma$ giving $P(v) = \frac{[v-(1+\sigma)]}{(v-\sigma)[v-(2+\sigma)]}$ and then follow the steps of Sec. 2.1. Thus,

Step 1: $n(v) = (v+1+\sigma)(v+z)$, $d(v) = (v+\sigma)(v+2+\sigma)$.

Step 2: $n(v) = [v^2 + z(1+\sigma)] + (1+\sigma+z)v$

$$d(v) = [v^2 + \sigma(2+\sigma)] + (2+2\sigma)v$$

Steps 3,4: $z(1+\sigma) = \sigma(2+\sigma)$

$$\frac{k_{max}}{k_{min}} = \left(\frac{2+2\sigma}{1+\sigma+z}\right)^2 = \frac{2(1+\sigma)^2}{2\sigma^2+4\sigma+1}$$

and $$PH(v) = kK\,\frac{[v^2-(1+\sigma)^2](v^2-z^2)}{(v^2-\sigma^2)[v^2-(2+\sigma)^2]}$$

Next replace v by $s+\sigma$, giving $PH(s) = \frac{kK(s-1)(s+2\sigma+1)(s+\sigma-z)(s+\sigma+z)}{s(s+2\sigma)(s-2)(s+2+2\sigma)}$ with root loci of $1+PH$ qualitatively shown in Fig. 3 for the case $\sigma = 1$, for which $z = 1.5$, $\frac{k_{max}}{k_{min}} = \rho = 1.3061$. At $(kK)_{min} = -1$ there is a pair of roots at -1 and another at $-1 \pm j\infty$. At $(kK)_{max} = -1.3061$, there is a double pair at $-1 \pm j\sqrt{3}$ (i.e. at $v = \pm j\sqrt{\sigma(2+\sigma)}$ or $s = -1 \pm j\sqrt{\sigma(2+\sigma)}$). Note that for $k \in [k_{min}, k_{max}]$, the system poles lie on the line $\sigma = -1$, but the system is stable for the larger k range of 1.453.

3.2 Far-off poles

In the optimum blending solution, as noted PH is finite as $s \to \infty$, which is acceptable. But it is not acceptable in the control problem where one should ensure $H \to 0$ as $s \to \infty$. In the above example it is therefore necessary to add at least two poles to H. If these are far-off, they will have little effect on the previous "optimum" results. For example, if they are inserted quite close in at 29, -31 the symmetry is preserved but ρ decreases to 1.164 for the loci to lie on the line $s = -1$. If the far-off poles are inserted at 99, -101 $\rho = 13.06/10.35 = 1.26$ for the poles on $s = -1$, and 1.41 for stability. The root loci are sketched in Fig. 4a. The further off the far-off poles the closer ρ is to the supremum of 1.306. For some range of k the complex pole damping factors are quite small. This can be remedied by using larger σ at the expense of smaller ρ. Another method is presented in Sec. 4.

The root loci for the "optimum" design for the same problem by (Bongiorno and Youla 1977) is shown in Fig. 4b. Here ρ(stability) is only 1.18 and $\rho = 1.10$ for the poles to be left of the line $s = -.4$. In addition their compensation $\to 218$ as $s \to \infty$, which is of course impractical, with a far-off pole at -203. The mandatory addition of at least one more pole would significantly reduce ρ unless it is extremely far-off. Note that this design technique has no provision for controlling the system poles over some uncertainty range, and the design procedure is much more complicated.

3.2 <u>The far-off pole problem</u>

In Example 2 of Sec. 3.1, the far-off pole problem was easily solved, but the same approach may completely spoil the design in other cases, as in

<u>Example 3</u>, where $P(s) = \frac{k(s-2)}{s-1}$. Following Sec. 3.1, $n(v) = v+2+\sigma$, $d(v) = v+1+\sigma$, so $\rho_{max} = \left(\frac{2+\sigma}{1+\sigma}\right)^2$. Note how ρ decreases with σ, which is the unsurprising result in general, whereby pole damping and $|S|$ peaking is traded against ρ. $G(v) = \frac{K(v+2+\sigma)}{v+1+\sigma} = K\frac{(v+3)}{(v+2)}$ for $\sigma = 1$ and $G(s) = K\left(\frac{s+4}{s+3}\right)$. Suppose far-off poles are added as in Sec. 3.1, say at $-p_1$, $-p_2$, then it is easily found that the closed-loop system is unstable $\forall\ p_1 > 2$, $p_2 > 0$ for K positive or negative and of any magnitude. But it is easy to insert lhp far-off poles and have a stable system over a significant k range, e,g, one of 1.85 for a pair of poles at -30 and larger range if they are further off. On the other hand, the latter approach may spoil the design as in

<u>Example 4</u>, where $P(s) = \frac{k(s-1)}{(s-2)}$. Following Sec. 3.1, $G(s) = \frac{K(s+3)}{(s+4)}$ if $\sigma = 1$. But the addition of even one lhp pole, no matter how far-off, gives an unstable system. However a pair of far-off poles at p_1, $-p_2$ may be inserted as in Example 2, giving a stable design.

The above examples reveal the importance of understanding the phenomenon and of finding a simple technique to determine how to proceed with the additional poles needed to ensure a practical design. To understand it, simply make an approximate qualitative Nyquist sketch of the 'optimum' but impractical $GP(j\omega)$ as is done in Figs. 5a-c for Examples 2-4. It is known, of course, that the 'optimum' design is stable so the number of counterclockwise -1 encirclements must equal the number of rhp poles of GP. This makes it very easy to make the qualitative Nyquist sketches, as one only needs to know whether to begin (at $GP(0)$) to the left or the right of -1. (In Fig. 5a the pole at the origin is counted in the right half-plane and the rhp boundary is indented to its left as shown in the insert in Fig. 5a, for the sake of consistency.) In Figs. 5a,c the Nyquist locus terminates at the left of -1, so lhp poles, which cause $GP \to 0$ as $\omega \to \infty$ and which contribute phase lag, must modify the locus as shown by the dashed lines—upsetting the encirclement count, no matter how far-off the poles. In Fig. 5b the dashed modification is all right providing the lhp poles are sufficiently far-off. However, in Figs. 5a,c an additional rhp pole properly placed gives one more negative half encirclement, as shown by the dotted lines, so that the point $GP(\infty)$ is at the right of -1, permitting thereafter more far-off lhp poles, if desired.

Thus, the simple rule is that the number of rhp poles must be such that $GP(\infty)$ is at the right of -1. Only then can far-off lhp poles be properly added. They can then be added in a manner very similar to open-loop stable designs — so as to have little effect on the gain and phase margins in the crossover regions. The plural is necessary here because there are as many crossovers as rhp poles.

If our 'optimum' design places $GP(\infty)$ to the left of -1 there is no choice but to insert an odd number of additional rhp poles (or remove an odd

number if possible), before any lhp far-off poles can be added. Such additional rhp poles can always be inserted sufficiently far-off to have as-small-as-desired effect on the $\rho = k_{max}/k_{min}$ of the original 'impractical optimum' design. If one wishes, they can be put closer in, considered as part of $P(s)$ and an 'impractical optimum design' found for this new $P(s)$, whose $PG(\infty)$ is now on the right of -1, permitting any finite number of additional lhp poles, if properly placed. However, such added rhp poles always decrease the ρ of the previous optimum design, the amount of decrease being smaller the further off it is. Note that in Example 2 of Sec. 2.1, a symmetrical pole pair was satisfactory. Additional lhp far-off poles may be then added, which is not possible without such or other preliminary change in the rhp.

3.3 The problem of highly underdamped closed-loop poles

In the design philosophy of this section, the optimum design maximizes $\rho = k_{max}/k_{min}$ for which all the closed-loop poles lie in $\mathcal{R}e\, s \leq -\sigma$. In this optimum design, there is a permissible k value for a system pole at each point on the line $s = -\sigma$ from $j0$ to $j\infty$, so that for some k range one or more closed loop complex pole pairs will be very highly underdamped — at the higher natural frequencies, of course. This may be intolerable. One remedy is to decrease the permissible k range. One might consider insertion of a complex pole or zero pair suitably on the line $\mathcal{R}e\, s = -\sigma$, in order to focibly curtail the root loci of $1+GP$. Thus in Example 3 of Sec. 3.2 with $\rho = 1$, consider use of $PG(v) = \dfrac{-kK(v^2-9)}{(v^2-4)(v^2+50)}$, thereby sacrificing "optimality" to some extent. It is easily found that for $kK \in [22.2, 29.65]$ with $\rho = 1.336$, the roots are on $s = -1$ from $-1 \pm j0$ to $-1 \pm j4.88$ with a minimum damping factor $\zeta = .2$. However, if the 'optimum' design is used, $\zeta_{min} = .48$ for the same value of ρ. Of course in the "sub-optimum" design

two relatively close-in additional poles have been obtained, relieving considerably or completely the far-off pole problem.

Highly underdamped complex poles in the ρ range can also be avoided by using circles, as in Sec. 4, instead of vertical lines as the boundary of the forbidden region. However, no matter how one squirms, the sensitivity function $S_P^T(j\omega) = (1+L)^{-1}$ tends to be large in the control bandwidth range, because of the need, noted in Sec. 1.2, of $L(j\omega)$ to negatively encircle -1 a sufficient number of times.

3.4 Elements in the Feedback Return Path

In Fig. 1, the return path transmission from C to U is -1, implying the sensor has infinite bandwidth with value 1. Its actual transfer function $M(s)$ is easily accommodated in the design procedure which emerges with $F(s)$, $G(s)$ in order to realize a $T(s)$, $S(s)$ pair. Suppose $M(s)$ is mp and stable. Then the same T, S are achieved by using F^*, G^* with $T = \frac{FGP}{1+GP} = \frac{F^*G^*P}{1+G^*PM}$, $S^{-1} = (1+GP) = (1+G^*PM)$, Hence, set $G^* = G/M$, $F^* = FG/G^* = FM$. If $M(s)$ has any rhp poles and/or zeros, they must be considered in the design procedure by including them explicitly in the loop transmission, exactly as those of $P(s)$. It must also be recognized that the rhp poles of M must appear as zeros of $T(s)$. Hence, if $T(s)$ is explicitly formulated, these rhp poles must be included, as well as the rhp zeros of $P(s)$. In our design procedure there is no explicit formulation of $S(s)$. The constraints on $S(s)$ due to the rhp poles and zeros of P, M are automatically handled by the design procedure, so long as they are explicitly included in $L(s)$.

4. CIRCLES AS LOCI OF CLOSED-LOOP POLES

The interior of a circle in the lhp instead of the region $\mathcal{R}e\, s \le -\sigma$, may be used as the acceptable region for the closed-loop poles, over some finite plant gain factor range. This gives an optimum design in which all the poles lie on the circle boundary. The design technique is just as simple as in Sec. 3. The transformation

$$w = \frac{s+b-r}{s+b+r} \quad , \quad s = \frac{b-r-w(b+r)}{w-1} \tag{8a,b}$$

maps the w-rhp plane into the interior of the circle of radius r , centered at $-b$ in the s plane. Hence to design, one first maps the rhp poles and zeros of $P(s)$ into the w plane, designs in the w plane in the manner of the blending problem of Sec. 3, and then maps back into the s plane. This is illustrated by several examples.

Example 3 (of Sec. 3.2)

$P = \dfrac{k(s-2)}{(s-1)}$ and say $r=1$, $b=2$ so $P_w = \dfrac{k'(w-.6)}{(w-.5)}$ and obviously $L_w = -k'K_1 \dfrac{(w^2-.36)}{(w^2-.25)}$ which becomes $L(s) = \dfrac{-kK(s-2)(s+1.75)}{(s-1)(s+\frac{5}{3})}$ with $\rho = (\frac{.6}{.5})^2 = 1.44$ fot the roots to lie on the circle. Note that in Sec. 3.2 $\rho = (\frac{3}{2})^2 = 2.25 > 1.44$. However, here the closed-loop poles are well damped for a larger range — from $kK = 10/21$ at which both poles are on the negative real axis to $kK = 1$ at which one root reaches $-\infty$. It follows from the discussion in Sec. 3.2 that lhp far-off poles may be added, with no rhp modification needed.

Example 2 (Sec. 3.1) with $b=2$, $r=1$

$P = \dfrac{k(s-1)}{s(s-2)}$ becomes $P_w = \dfrac{k'(w-.5)}{(w-\frac{1}{3})(w-.6)}$. In the manner of Sec. 3.1 write $n(w) = (w+.5)(w+m) = (w^2+.5m) + (.5+m)w$, $d(w) = (w+\frac{1}{3})(w+.6) = (w^2+.2) + .933w$. Hence, $.5m = .2$ and $\rho = [.933/(.5+m)]^2 = 1.075$ for the

roots to lie on the circle. The compensation G_w has zeros at $\pm.4$ which map in the s plane (Eq. 8b) as zeros of $G(s)$ at 1/3, -11/7. The far-off pole situation is unaltered, i.e. it is necessary to add at least one more rhp pole. To do this, let $P_w = \frac{k'(w-.5)}{(w-\frac{1}{3})(w-.6)(w-A)}$ so in the procedure of Sec. 3.1, $n(w) = (w+.5)(w^2+Mw+N)$, $d(w) = (w+\frac{1}{3})(w+.6)(w+A)$. Expanding each into even and odd polynomials and equating coefficients etc., gives (for $A=100$) $M=104.134$, $N=41.466$. Transforming back to the s plane, gives

$$L(s) = \frac{-kK(s-1)(s+\frac{5}{3})(s+2.9809)(s+3.0195)(s+1.5711)(s-.33167)}{s(s+1.5)(s-2)(s+1.75)(s+3.0202)(s+2.9802)}$$

with the root loci of $1+L_s$ shown in Fig. 6a. The new value of ρ is almost exactly the same as the previous, because A was taken so large. It is now possible to add lhp far-off poles to $L(s)$. Note the reduction in ρ but in return the system poles are very well damped over this range. It is found that $|S(j\omega)|_{max}$ is 15.9db (for $kK=1$), compared to a peak value of 24.6db in the optimum (Bongiorno and Youla 1977) design. Thus, this approach gives the designer a flexible means of trade-off between ρ and $|S(j\omega)|_{max}$.

<u>Example 5 with $b=7$, $r=6$</u>

$P = \frac{k(s-2)}{s(s-1)}$ and a more conservative circle is used (Fig. 6b). Eqs. (8a,b) become $w = \frac{s+1}{s+13}$, $s = \frac{1-13w}{w-1}$ so the zero and poles of $P(s)$ map into .2, 1/13, 1/7 in the w plane. A rhp pole at A is added to $P(s)$, to handle the far-off pole problem. Following the technique of Sec. 3.1,

$$\begin{aligned} n(s) &= (w+.2)(w^2+Mw+B) \\ &= w^3 + w(B+.2M) + (.2+M)(w^2+\frac{.2B}{.2+M}) \end{aligned}$$

$$\begin{aligned} d(s) &= (w+\frac{1}{13})(w+\frac{1}{7})(w+A) \\ &= w^3 + w(.011+.2198A) + (.2198+A)(w^2+\frac{.011A}{.2198+A}) \end{aligned}$$

Let $B + .2M = .011 + .22A$, $\frac{.2B}{.2+M} = \frac{.011A}{.22+A}$, giving (for $A = 100$) $B = 4.7398$, $M = 86.246$, $\rho = (\frac{.2198 + A}{.2 + M})^2 = 1.344$ for the loci to remain on the circle. Mapping back into the s-plane gives a compensation network $G(s)$ with zeros at -12.8624, -13.1409, -1.6255, -.3017, -3 and poles at -13.1212, -12.8812, -1.85714, -2.5. Left half-plane far-off poles may now be added. The root loci of $1+L$ are sketched in Fig. 6b.

5. INTERLACING PROPERTY OF RHP REAL POLES AND ZEROS

It is noted that in all the examples, the loop transmission $L(s)$ which emerged had an even number of $\binom{\text{zeros}}{\text{poles}}$ between any two $\binom{\text{poles}}{\text{zeros}}$ on the positive real axis. This necessary and sufficient condition for system design was previously noted by (Youla et al 1974). It is not explicitly needed in our design approach as it automatically emerges. Nevertheless, it is worth noting and presenting herewith a very simple proof of its necessity. Its sufficiency is obvious from the constructive nature of the design technique.

Let $L = \frac{KN(s)f(s)}{D(s)g(s)}$, so that

$$\text{Num.}(1+L) = Dg + KNf = \varphi(s) \quad , \tag{9}$$

where $\varphi(s)$ and the monic polynomials D , N , have no positive real zeros and the monic polynomials f , g have only positive real zeros. Consider any two consecutive zeros of $g(s)$, say at p_i , $p_j > 0$. From (9), $\varphi(p_i) = KNf(p_i)$ and $\varphi(p_j) = KNf(p_j)$. Since $N(s)$ has no positive real zeros and K is a constant, the difference in signs of $\varphi(p_i)$ and $\varphi(p_j)$ is precisely equal to the difference in signs of $f(p_i)$, $f(p_j)$. There is an alteration in the latter iff $f(s)$ has an odd number of zeros between p_i , p_j . But since $\varphi(s)$ has no positive real zeros, there can be no

difference in signs of $\varphi(p_i)$ and $\varphi(p_j)$. Hence, $f(s)$ must have an even number of zeros between p_i , p_j . A similar condition for the zeros of $g(s)$ is proven in the same manner.

Suppose $L(s)$ has an excess $e_L > 0$ of poles over zeros, so that as $s \to \infty$, $\varphi(s) \to Dg(s) > 0$ because D , g and in this case φ , are monic polynomials. Consider the largest positive real zero of $f(s)$ say at z , so $D(z)g(z) = \varphi(z)$ and the sign of KNf is the same for s real $> z$. Let $g(s)$ have m positive zeros in $[z,\infty)$ with the largest at p . Hence, $\text{sgn}\, Dg(s)$ (for s real $> p$) $= (-1)^m \, \text{sgn}\, D(z)g(z) = (-1)^m \, \text{sgn}\, \varphi(z)$. But $\text{sgn}\, Dg(s) = \text{sign}\, \varphi(s) > 0$ for $s > p$ because $Dg \to \varphi$ as $s \to \infty$ and $\varphi(s)$ is monic. Hence m must be even.

6. CONCLUSIONS

A very simple, straightforward design procedure has been presented for feedback systems with constrained rhp poles and zeros. The inherent limitations in the sensitivity reduction properties, which are difficult to cope with in the frequency domain, are handled by a design which restricts the closed-loop poles to certain specific regions in the complex plane. The design is optimum in the sense of giving the maximum gain-factor uncertainty for which such restrictions are satisfied. The most significant shortcoming of the design technique is that uncertainty in plant poles and zeros is not considered.

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Symbols

lhp	left half-plane
mp	minimum-phase
nmp	nonminimum-phase
odf	one-degree-of-freedom
rhp	right half-plane
tdf	two-degree-of-freedom

Figure Titles

Fig. 1. Two-degree-of-freedom structure

$$L = PG \quad , \quad T(s) = \frac{C(s)}{R(s)} = F\,\frac{L}{1+L} \quad (\ F = 1 \ \text{ in one-degree-of-freedom structure})$$

Fig. 2. Example 1 from Blending Problem — Root Loci of $1+L=0$.

$$L = \frac{-10^4 k(s^2-1)(s^2-16)(s^2-(.0276)^2)(s^2 \pm 1.811s + 5.902)}{(s^2-4)(s^2-9)(s^2-(0.239)^2)(s^2 \pm s + 4)}$$

Fig. 3. Example 2 — Root Loci of $1 - \dfrac{k(s-1)(s-.5)(s+2.5)(s+3)}{s(s^2-4)(s+4)}$

Fig. 4. Comparisons of designs; root loci of $1+L=0$.

(a) $L = \dfrac{-kK(s-.5)(s-1)(s+2.5)(s+3)}{s(s^2-4)(s+4)(s-99)(s+101)}$

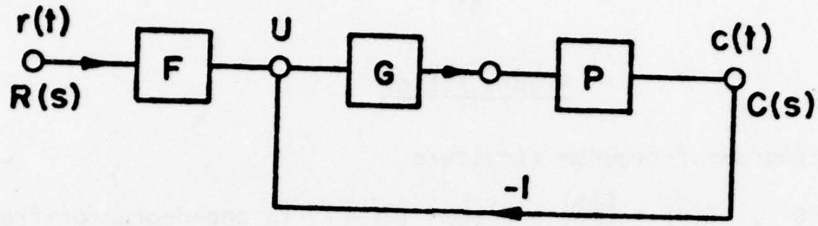
(b) $L = \dfrac{-k(s-1)(218s^3+3938s^2-2200s-1000)}{s(s-2)(s^3-185s^2-3759s-50)}$

Fig. 5. Qualitative Nyquist sketches (a) Example 2 (b) Ex. 3 (c) Ex. 4. Dashed lines show effect of lhp far-off poles. Dotted lines in (a,c) show effect of additional rhp pole.

Fig. 6. Root loci for designs with roots on circle

(a) $P = \dfrac{k(s-1)}{s(s-2)}$; centre at -2, radius 1.

(b) $P = \dfrac{k(s-2)}{s(s-1)}$; centre at -7, radius 6.



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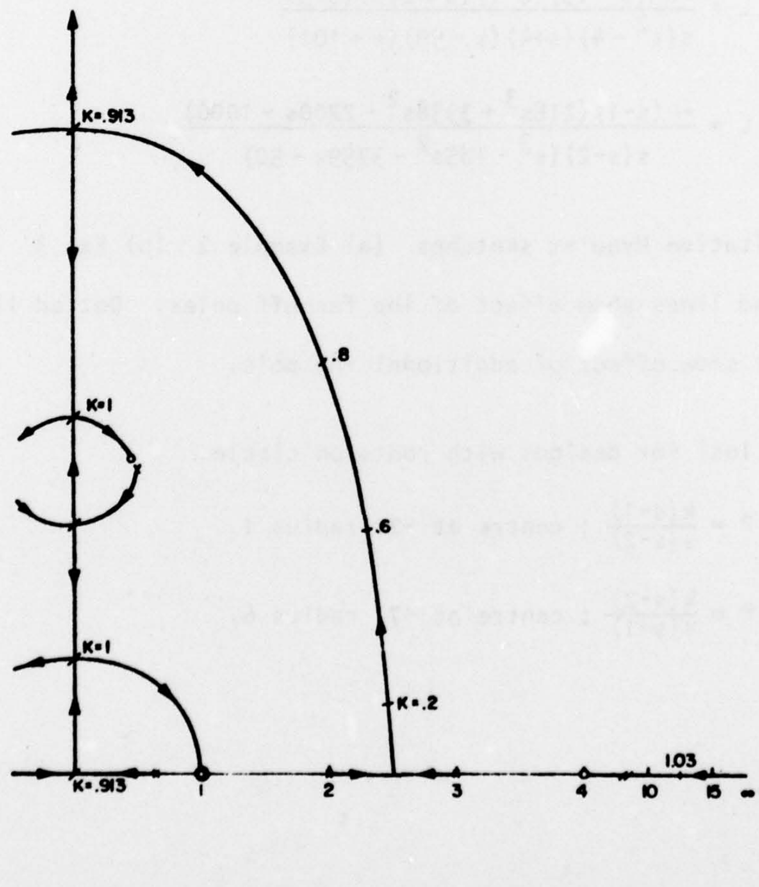

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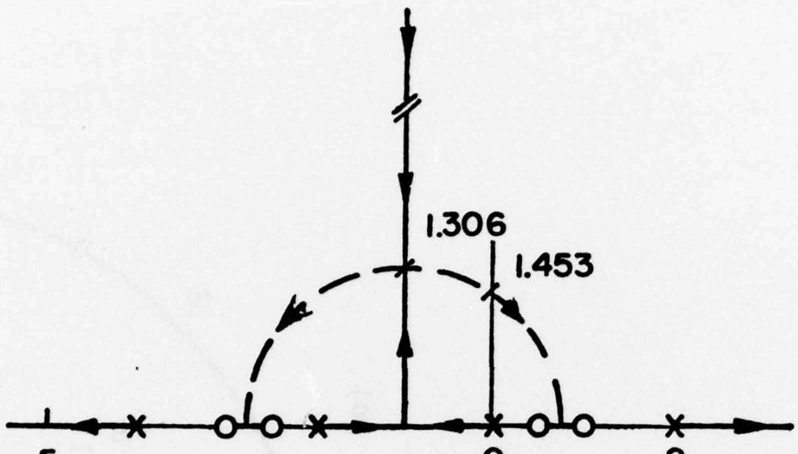

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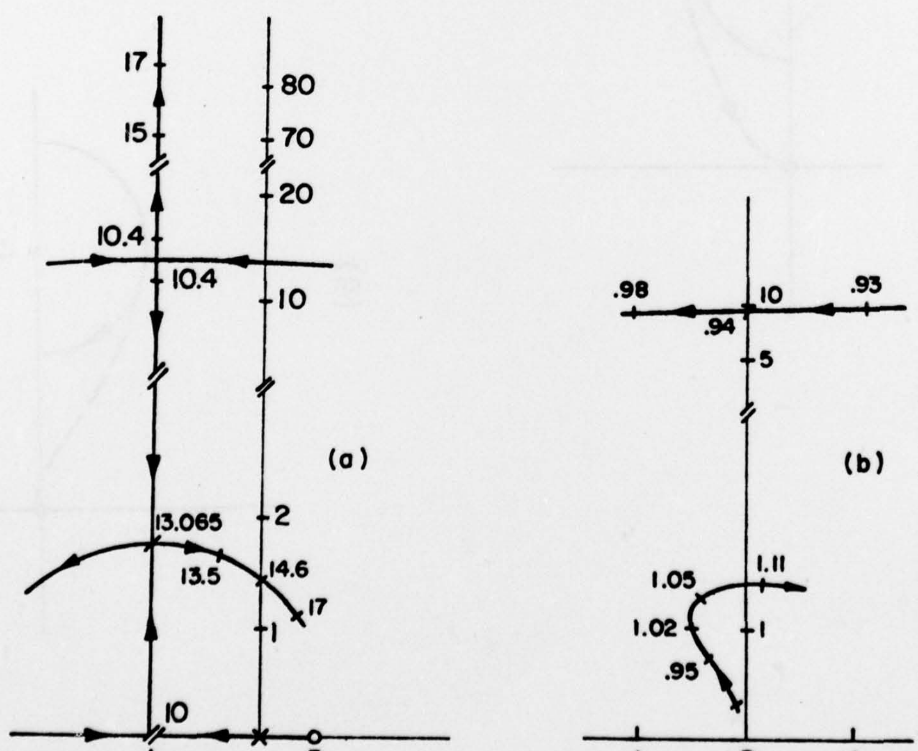

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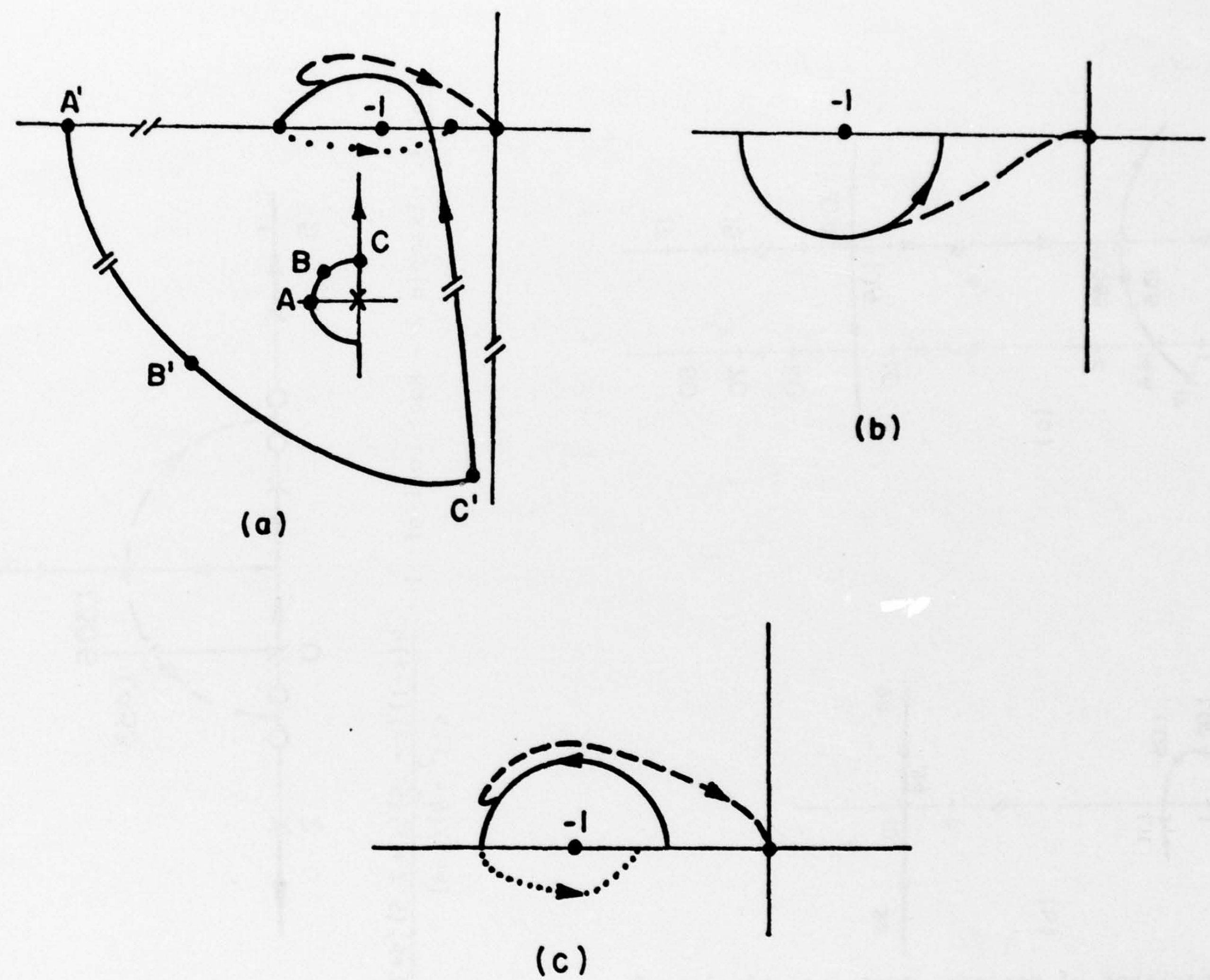

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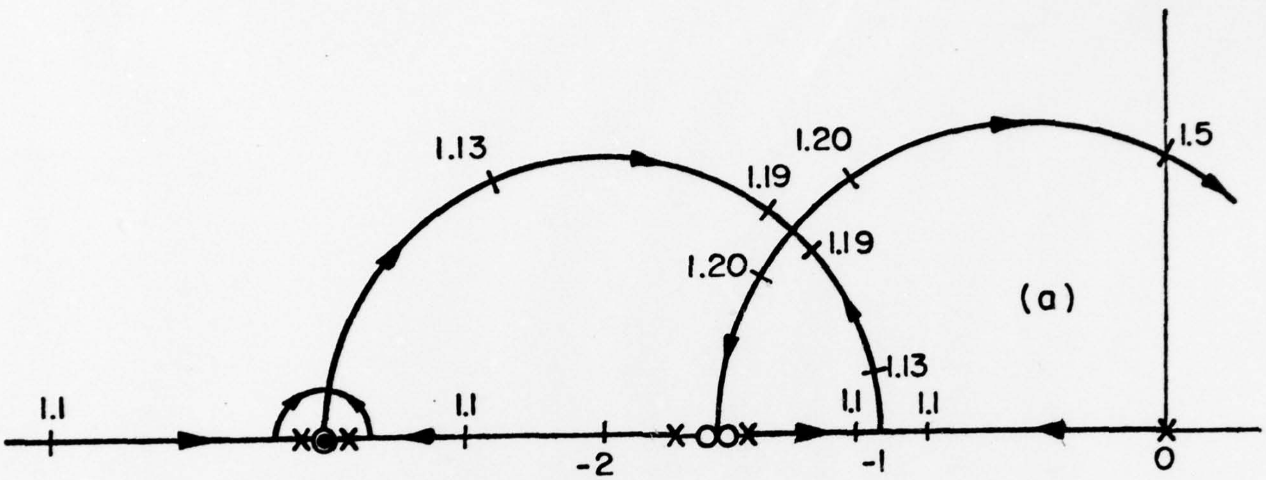

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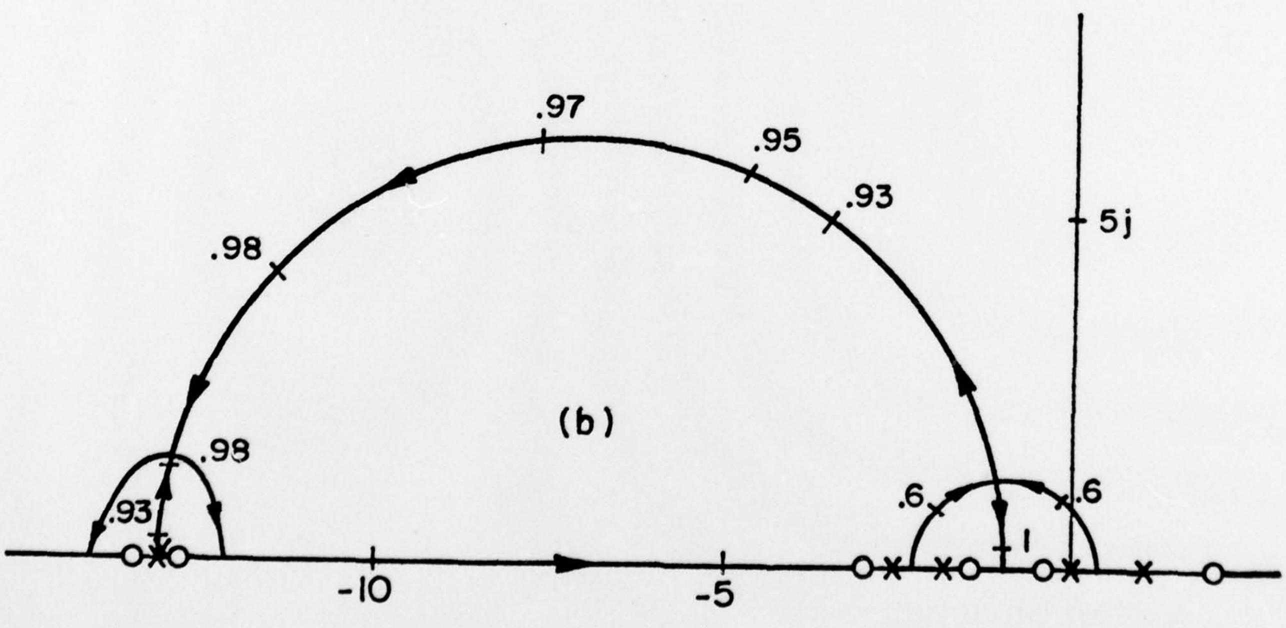

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Feedback systems with right half-plane poles and zeros may have inherently very poor sensitivity properties. In the design procedure presented, the closed-loop poles are restricted to two possible regions in the complex plane. One region is $\mathcal{R}e\ s \leq -\sigma$, $\sigma > 0$. A second is the interior and boundary of a circle in the left half-plane. The design is optimum in the sense of maximizing the gain factor uncertainty, for which the restriction is satisfied. The design procedure is very simple to execute and results in loop transmission poles and zeros which are

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20. Abstract continued

symmetrical with respect to the boundary of the forbidden region. The closed-loop poles lie entirely on the boundary, over the range of gain uncertainty.